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Limit Behavior of Insurance Premiums Dependent on Parameters

Research Article

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Abstract: Several limit theorems describing convergence of the insurance premiums to the essential supremum of insurance losses while premium parameters grow to plus infinity are presented. Mentioned theorems cover cases of exponential, risk adjusted, Esscher, and proportional hazard transform premium calculation principles. Sufficient condition of convergence of the exponential premium to the net premium while premium's parameter tends to zero from above is also presented.

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1. Introduction

Let us consider a random variable X, with distribution function $F_X(x)$, representing size of the insurance compensation related to a particular insurance pact. Premium to be paid for the risk X will be denoted as $\pi[X]$. In majority of the cases the random variable X is assumed to be a non-negative one, i.e., it takes vale zero if the contract will not produce a claim and will be equal to the claim size if there will be a claim. In some cases, however, negative values of the variable X are also allowed; such negative values are often interpreted as compensations which have to be paid by the customer to the insurance company, for example, as penalties for violation of the contract conditions. Let us now define several insurance premium calculation principles which we would like to investigate.

Net premium for an arbitrary risk X, which in the article will be denoted by $\pi_{\text{net}}[X]$, is defined as the expected value of the losses associated with the risk X, i.e.,

$$\pi_{\text{net}}[X] \ := \ \mathsf{E}[X].$$

Exponential premium, dependent on a parameter α, for an arbitrary risk X, which in the article will be denoted by $\pi_{\text{exp.}(\alpha)}[X]$, is defined in the following way

$$\pi_{\text{exp.}(\alpha)}[X] \ := \ \frac{1}{\alpha} \log \left(\mathsf{E}[e^{\alpha X}] \right), \quad \text{for} \quad \alpha > 0.$$

Risk adjusted premium, dependent on a parameter ρ, for a non-negative risk X with distribution function $F_X(x)$, which in the article will be denoted as $\pi_{\text{r.a.}(\rho)}[X]$, is defined in the following way

$$\pi_{\text{r.a.}(\rho)}[X] \ := \ \int_0^\infty [1 - F_X(x)]^{1/\rho} dx, \quad \text{for} \quad \rho \geq 1.$$

Proportional hazard transform premium, dependent on a parameter ρ, for an arbitrary risk X with distribution function $F_X(x)$, which in the article will be denoted as $\pi_{\text{p.h.t.}(\rho)}[X]$, is defined as follows

$$\pi_{\text{p.h.t.}(\rho)}[X] \;:=\; \int_0^{\infty} [1 - F_X(x)]^{1/\rho} dx - \int_{-\infty}^{0} 1 - [1 - F_X(x)]^{1/\rho} dx, \quad \text{for} \quad \rho \geq 1.$$

Esscher premium, dependent on a parameter α, for an arbitrary risk X, which in the article will be denoted as $\pi_{\text{Esscher}(\alpha)}[X]$, is defined in the following way

$$\pi_{\text{Esscher}(\alpha)}[X] \;:=\; \frac{\mathsf{E}[Xe^{\alpha X}]}{\mathsf{E}[e^{\alpha X}]}, \quad \text{for} \quad \alpha \geq 0.$$

We will also need the following two definitions.

Essential supremum of an arbitrary random variable X with distribution function $F_X(x)$ is defined as follows

$$\text{ess sup}[X] \;:=\; \sup\{\delta : \, F_X(\delta) < 1\}.$$

Essential infimum of an arbitrary random variable X with distribution function $F_X(x)$ is defined as follows

$$\text{ess inf}[X] \;:=\; \inf\{\delta : \, F_X(\delta) > 0\}.$$

Let us remind that the essential supremum of a random variable X in the actuarial literature is often referred as the *maximal loss premium* associated with the risk X.

More information about the above defined methods of pricing of insurance contracts as well as desirable properties that can be possessed or not possessed by an insurance premium calculation principle can be found, for example, in Asmussen and Albrecher (2010), Boland (2007), Bowers et al (1997), Bühlmann (1970), Dickson (2005), Gerber (1979), De Vylder et al (1984), De Vylder et al (1986), Kaas et al (2008), Kremer (1999), Rolski et al (1998), Straub (1988).

2. Exponential Premium Principle

The following theorem illustrates sufficient condition of convergence of the exponential premium to the net premium while the premium parameter tends to zero from above.

Theorem 2.1. *If for a risk X there exist $\varepsilon > 0$ such that $\mathsf{E}[|Xe^{\varepsilon X}|] < +\infty$, then*

$$\lim_{\alpha \to 0_+} \pi_{\text{exp.}(\alpha)}[X] \;=\; \pi_{\text{net}}[X].$$

Proof. Let us show first that fulfilment of the condition $\mathsf{E}[|Xe^{\varepsilon X}|] < +\infty$ implies also fulfilment of the condition $\mathsf{E}[e^{\varepsilon X}] < +\infty$. Indeed, using inequalities

$$\mathsf{E}[\mathbf{1}_{\{|X|\geq 1\}}|Xe^{\varepsilon X}|] \;\leq\; \mathsf{E}[|Xe^{\varepsilon X}|] \;<\; +\infty,$$

obtain

$$\mathsf{E}[e^{\varepsilon X}] \;=\; \mathsf{E}[\mathbf{1}_{\{|X|<1\}}\, e^{\varepsilon X}] + \mathsf{E}[\mathbf{1}_{\{|X|\geq 1\}}\, e^{\varepsilon X}] \;\leq\; e^{\varepsilon} + \mathsf{E}[\mathbf{1}_{\{|X|\geq 1\}}|Xe^{\varepsilon X}|] \;<\; +\infty.$$

Condition $\mathsf{E}[e^{\varepsilon X}] < +\infty$ is sufficient for changing of the order of differentiation and integration of the power series (1) in the neighborhood of the origin

$$\frac{d}{d\alpha}\mathsf{E}[e^{\alpha X}]\bigg|_{\alpha=0} \;=\; \frac{d}{d\alpha}\mathsf{E}\left[\sum_{k=0}^{+\infty}\frac{(\alpha X)^k}{k!}\right]\bigg|_{\alpha=0} \;=\; \mathsf{E}\left[\frac{d}{d\alpha}\sum_{k=0}^{+\infty}\frac{(\alpha X)^k}{k!}\right]\bigg|_{\alpha=0} \;=\; \sum_{k=0}^{+\infty}\frac{\alpha^k \mathsf{E}[X^{k+1}]}{k!}\bigg|_{\alpha=0} \;=\; \mathsf{E}[X]. \tag{1}$$

It is easy to see that for any risk X the following two identities hold

$$\mathsf{E}[e^{\alpha X}]\Big|_{\alpha=0} \;=\; 1 \qquad \text{and} \qquad \log\left(\mathsf{E}[e^{\alpha X}]\Big|_{\alpha=0}\right) \;=\; 0. \tag{2}$$

Observe that fulfilment of the condition $\mathsf{E}[|Xe^{\varepsilon X}|] < +\infty$, for some fixed $\varepsilon > 0$, is sufficient for differentiability of the function $\mathsf{E}[e^{\alpha X}]$ in the neighborhood of origin, and the function $\log(\mathsf{E}[e^{\alpha X}])$ therefore will be differentiable as a superposition of differentiable functions. Hence, using identities (1) and (2), we get

$$\lim_{\alpha\to 0_+} \pi_{\exp.(\alpha)}[X] \;=\; \lim_{\alpha\to 0_+} \frac{\log(\mathsf{E}[e^{\alpha X}]) - \log(\mathsf{E}[e^{\alpha X}]|_{\alpha=0})}{\alpha} \;=\; \frac{d}{d\alpha}\log(\mathsf{E}[e^{\alpha X}])\Big|_{\alpha=0} \;=\; \frac{d}{d\alpha}\mathsf{E}[e^{\alpha X}]\Big|_{\alpha=0} : \mathsf{E}[e^{\alpha X}]\Big|_{\alpha=0} \;=\; \mathsf{E}[X].$$

This completes the proof of Theorem 2.1. □

The following theorem illustrates convergence of the exponential premium to the essential supremum of the priced risk as the premium parameter tends to plus infinity.

Theorem 2.2. *For any risk X holds limit relation*

$$\lim_{\alpha\to+\infty} \pi_{\exp.(\alpha)}[X] \;=\; \text{ess sup}[X]. \tag{3}$$

Proof. Let us show first that for any risk X and any admissible value of the parameter α the exponential premium for the risk X will not exceed essential supremum of the risk X, or in other words

$$\pi_{\exp.(\alpha)}[X] \;\le\; \text{ess sup}[X]. \tag{4}$$

Observe that it is enough to demonstrate fulfilment of inequality (4) in the case of ess $\sup[X] < +\infty$, because otherwise inequality (4) holds automatically. So here we get

$$\pi_{\exp.(\alpha)}[X] \;=\; \frac{1}{\alpha}\log(\mathsf{E}[e^{\alpha X}]) \;\le\; \frac{1}{\alpha}\log(\mathsf{E}[e^{\alpha \text{ ess sup}[X]}]) \;=\; \text{ess sup}[X]. \tag{5}$$

It is easy to see that for any risk X and any real constant c such that $c <$ ess $\sup[X]$ the following inequalities hold

$$\mathsf{P}\{X \ge c\} \;>\; 0 \qquad \text{and} \qquad \mathsf{E}[e^{\alpha X}] \;\ge\; e^{\alpha c}\mathsf{P}\{X \ge c\}. \tag{6}$$

Using inequalities (6), we get

$$\lim_{\alpha\to+\infty} \frac{1}{\alpha}\log(\mathsf{E}[e^{\alpha X}]) \;\ge\; \lim_{\alpha\to+\infty} \frac{1}{\alpha}\log(e^{\alpha c}\mathsf{P}\{X \ge c\}) \;=\; \lim_{\alpha\to+\infty} \frac{1}{\alpha}[\log(e^{\alpha c}) + \log(\mathsf{P}\{X \ge c\})] \;=\; \lim_{\alpha\to+\infty} \frac{1}{\alpha}[\alpha c] + 0 \;=\; c. \tag{7}$$

Combining limit form of inequality (4) with inequality (7) we finally get

$$\lim_{\alpha\to+\infty} \pi_{\exp.(\alpha)}[X] \;\in\; \bigcap_{c<\text{ess sup}[X]} [c,\ \text{ess sup}[X]] \;\equiv\; \{\text{ess sup}[X]\}.$$

This completes the proof of Theorem 2.2. □

From inequality (4) it follows that the exponential premium will converge to the essential supremum of the size of insurance compensation *from below* as the premium parameter tends to plus infinity. Non-negativity of the risk X was not used within the proof of Theorem 2.2, therefore limit relation (3) holds for any random variable X and not only for a non-negative one. Due to some economical reasonings, exponential premium for a risk X was defined only for the strictly positive values of the parameter α, however it is interesting to see that for any random variable X holds also the following limit relation

$$\lim_{\beta\to-\infty}\frac{1}{\beta}\log(\mathsf{E}[e^{\beta X}]) \;=\; \text{ess inf}[X]. \tag{8}$$

Indeed, using statement of Theorem 2.2, for any random variable X obtain

$$\lim_{\beta\to-\infty}\frac{1}{\beta}\log(\mathsf{E}[e^{\beta X}]) \;=\; -\lim_{\beta\to-\infty}\frac{1}{-\beta}\log(\mathsf{E}[e^{-\beta(-X)}]) \;=\; -\lim_{\alpha\to\infty}\frac{1}{\alpha}\log(\mathsf{E}[e^{\alpha(-X)}]) \;=\; -\text{ess sup}[-X] \;=\; \text{ess inf}[X],$$

therefore, for arbitrary choice of a variable X limit relation (8) holds.

3. Risk Adjusted Premium Principle

The following theorem demonstrates convergence of the risk adjusted premium for a non-negative risk to essential supremum of size of the insurance compensation when the premium parameter tends to plus infinity.

Theorem 3.1. *For any non-negative risk X holds limit relation*

$$\lim_{\rho\to+\infty}\pi_{\text{r.a.}(\rho)}[X] \;=\; \text{ess sup}[X]. \tag{9}$$

Proof. Let us show first that for any non-negative risk X and any admissible value of the parameter ρ, the risk adjusted premium will not exceed essential supreme of size of the insurance compensation, i.e.,

$$\pi_{\text{r.a.}(\rho)}[X] \;\le\; \text{ess sup}[X]. \tag{10}$$

Indeed, using inequalities $0 \le [1-F_X(x)]^{1/\rho} \le 1$, for $x\in\mathbb{R}$ and $\rho\ge 1$, obtain

$$\pi_{\text{r.a.}(\rho)}[X] \;=\; \int_0^{+\infty}[1-F_X(x)]^{1/\rho}dx \;\le\; \int_0^{\text{ess sup}[X]}dx \;=\; \text{ess sup}[X].$$

In the case of a degenerated non-negative risk X, i.e., in the case of $0\le \text{ess inf}[X] = \text{ess sup}[X]$, statement of Theorem 3.1 holds. Indeed, in the mentioned case we get

$$\lim_{\rho\to+\infty}\pi_{\text{r.a.}(\rho)}[X] \;=\; \lim_{\rho\to+\infty}\int_0^{\text{ess sup}[X]}[1-0]^{1/\rho}dx \;=\; \text{ess sup}[X].$$

Let us now demonstrate fulfilment of Theorem 3.1 in the case of a non-negative non-degenerated risk X, i.e., in the case of

$$0 \;\le\; \text{ess inf}[X] \;<\; \text{ess sup}[X].$$

Under described circumstances, for any real constant c, such that $\text{ess inf}[X] < c < \text{ess sup}[X]$, will hold inequalities

$$0 < [1-F_X(c)]^{1/\rho} < 1, \quad \text{for} \quad \rho\ge 1. \tag{11}$$

Taking into account non-decreasing structure of the function $F_X(x)$ and using inequalities (11), obtain

$$\pi_{\text{r.a.}(\rho)}[X] \ = \ \int_0^{+\infty} [1 - F_X(x)]^{1/\rho} dx \ > \ \int_0^{c} [1 - F_X(x)]^{1/\rho} dx \ > \ [1 - F_X(c)]^{1/\rho} \cdot c \ \to \ c, \quad \text{as} \quad \rho \to +\infty. \tag{12}$$

Combining limit form of inequality (10) with inequality (12) we finally get

$$\lim_{\rho \to +\infty} \pi_{\text{r.a.}(\rho)}[X] \ \in \bigcap_{\text{ess inf}[X] < c < \text{ess sup}[X]} [c, \ \text{ess sup}[X]] \ \equiv \ \{\text{ess sup}[X]\}.$$

This completes the proof of Theorem 3.1. $\square$

From inequality (10) it follows that the risk adjusted premium for a non-negative risk X will converge to essential supremum of the insurance compensation *from below* as the premium parameter tends to plus infinity.

4. Proportional Hazard Transform Premium Principle

The following theorem illustrates convergence of the proportional hazard transform premium to essential supremum of size of the insurance compensation when the premium parameter tends to plus infinity.

Theorem 4.1. *For any risk X holds limit relation*

$$\lim_{\rho \to +\infty} \pi_{\text{p.h.t.}(\rho)}[X] \ = \ \text{ess sup}[X]. \tag{13}$$

Proof. Let us show first that for any risk X and any admissible choice of the parameter ρ the proportional hazard transform premium will not exceed essential supremum of size of the insurance compensation, i.e.,

$$\pi_{\text{p.h.t.}(\rho)}[X] \ \le \ \text{ess sup}[X]. \tag{14}$$

Using inequalities $0 \le [1 - F_X(x)]^{1/\rho} \le 1$, for $x \in \mathbb{R}$ and $\rho \ge 1$, obtain

$$\pi_{\text{p.h.t.}(\rho)}[X] \ \le \ \int_0^{\max\{0,\, \text{ess sup}[X]\}} dx \ - \ \int_{\min\{0,\, \text{ess sup}[X]\}}^{0} dx \ =$$

$$= \ \max\{0, \ \text{ess sup}[X]\} \ + \ \min\{0, \ \text{ess sup}[X]\} \ = \ \text{ess sup}[X],$$

therefore, inequality (14) indeed holds. Let us show now that statement of Theorem 4.1 holds in the case of a degenerated risk X, i.e., in the case when $\mathsf{P}\{X = C\} = 1$, for some constant $C \in \mathbb{R}$. Here we get

$$\lim_{\rho \to +\infty} \pi_{\text{p.h.t.}(\rho)}[X] \ = \ \lim_{\rho \to +\infty} \left(\int_0^{\max\{0,\, C\}} [1 - 0]^{1/\rho} dx \ - \ \int_{\min\{0,\, C\}}^{0} 1 - [1 - 1]^{1/\rho} dx \right) \ = \ \max\{0, \, C\} + \min\{0, \, C\} \ = \ C,$$

moreover, since in the considered case $C = \text{ess sup}[X]$ then the statement of Theorem 4.1 indeed holds in the case of a degenerated risk. In the case of a non-degenerated risk X, i.e., in the case when ess $\inf[X] <$ ess $\sup[X]$, for any real constant c such that ess $\inf[X] < c <$ ess $\sup[X]$ will hold inequalities $0 < [1 - F_X(c)]^{1/\rho} < 1$, for $\rho \ge 1$. using which, taking into account non-decreasing structure of the function $F_X(x)$, as well as *uniform* limit relation

$$0 \ \le 1 - [1 - F_X(x)]^{1/\rho} \ \le \ 1 - [1 - F_X(\min\{0, \, c\})]^{1/\rho} \ \to 0, \quad \text{as} \quad \rho \to +\infty, \quad \text{for all} \quad x \le \min\{0, \, c\},$$

which implies the following

$$\lim_{\rho\to+\infty}\int_{-\infty}^{\min\{0,\,c\}} 1-[1-F_X(x)]^{1/\rho}dx \;=\; \int_{-\infty}^{\min\{0,\,c\}} \lim_{\rho\to+\infty}\left(1-[1-F_X(x)]^{1/\rho}\right)dx \;=\; 0,$$

obtain

$$\pi_{\text{p.h.t.}(\rho)}[X] \;>\; [1-F_X(c)]^{1/\rho}\cdot\max\{0,\,c\} \;-\; \int_{-\infty}^{\min\{0,\,c\}} 1-[1-F_X(x)]^{1/\rho}dx - \int_{\min\{0,\,c\}}^{0} dx \to\; \max\{0,\,c\}+\min\{0,\,c\} \;=\; c,$$

as $\rho\to+\infty$. Combining just obtained limit relation with the limit form of inequality (14) we finally get

$$\lim_{\rho\to+\infty}\pi_{\text{p.h.t.}(\rho)}[X] \;\in\; \bigcap_{\text{ess inf}[X]<c<\text{ess sup}[X]} [c,\;\text{ess sup}[X]] \;\equiv\; \{\text{ess sup}[X]\}.$$

This completes the proof of Theorem 4.1. □

From inequality (14) it follows that the proportional hazard transform premium will converge to essential supremum of size of the insurance compensation *from below* when the premium parameter tends to plus infinity.

5. Esscher Premium Principle

The next theorem illustrates convergence of the Esscher premium to essential supremum of size of the insurance compensation when the premium parameter tends to plus infinity.

Theorem 5.1. *For any risk X holds limit relation*

$$\lim_{\alpha\to+\infty}\pi_{\text{Esscher}(\alpha)}[X] \;=\; \text{ess sup}[X].$$

Let us from the beginning give a proof for Theorem 5.1 in the case of a non-negative risk X (non-negativity of the risk is a common assumption for majority of practical applications) since such proof might look technically somewhat simpler than the corresponding proof in the case of an arbitrary risk X.

Proof in the case of a non-negative risk. Let us show first that for any risk X and any admissible value of the parameter α the Esscher premium will not exceed ess sup$[X]$; it is enough to show this for ess sup$[X]<+\infty$ since otherwise the statement holds automatically. Here we get

$$\pi_{\text{Esscher}(\alpha)[X]} \;\le\; \frac{\mathsf{E}[\text{ess sup}[X]e^{\alpha X}]}{\mathsf{E}[e^{\alpha X}]} \;=\; \frac{\text{ess sup}[X]\cdot\mathsf{E}[e^{\alpha X}]}{\mathsf{E}[e^{\alpha X}]} \;=\; \text{ess sup}[X]. \tag{15}$$

In the case of a degenerated non-negative risk X (this is the case when $\mathsf{P}\{X=C\}=1$ for some $C\in\mathbb{R}_0^+$) obtain

$$\pi_{\text{Esscher}(\alpha)[X]} \;=\; \frac{\mathsf{E}[C\cdot e^{\alpha\cdot C}]}{\mathsf{E}[e^{\alpha\cdot C}]} \;=\; \frac{C\cdot\mathsf{E}[e^{\alpha\cdot C}]}{\mathsf{E}[e^{\alpha\cdot C}]} \;=\; C \;=\; \text{ess sup}[X], \tag{16}$$

switching to the limit in (16), we see that statement of Theorem 5.1 holds in the case of a degenerated non-negative risk. In order to make our notations a bit more compact, we will sometimes use abbreviations:

$$\underline{b}:=\text{ess inf}[X] \quad\text{and}\quad \overline{b}:=\text{ess sup}[X].$$

Let us now prove statement of Theorem 5.1 in the case of a non-degenerated non-negative risk. Under such circumstances, for any real positive constant c satisfying inequalities $0 \le \text{ess inf}[X] < c < \text{ess sup}[X]$ the Esscher premium can be represented as follows

$$\pi_{\text{Esscher}(\alpha)}[X] \; = \; \frac{\int_{\underline{b}}^{c} x e^{\alpha x} dF_X(x) + \int_{c}^{\overline{b}} x e^{\alpha x} dF_X(x)}{\int_{\underline{b}}^{c} e^{\alpha x} dF_X(x) + \int_{c}^{\overline{b}} e^{\alpha x} dF_X(x)} \; \ge \; \frac{\int_{c}^{\overline{b}} x e^{\alpha x} dF_X(x)}{\int_{\underline{b}}^{c} e^{\alpha x} dF_X(x) + \int_{c}^{\overline{b}} e^{\alpha x} dF_X(x)} \; =: \; \pi^{\text{estimate}(c)}_{\text{Esscher}(\alpha)}[X].$$

In such a way we create a lower bound estimate, based on a positive real constant c, for the Esscher premium. Such a lower bound estimate will help us to make some conclusions related to the limit behavior of the Esscher premium itself. Let us show the fulfillment of the following limit relation

$$\int_{\underline{b}}^{c} e^{\alpha x} dF_X(x) \; = \; o\left(\int_{c}^{\overline{b}} e^{\alpha x} dF_X(x)\right), \quad \text{as} \quad \alpha \to +\infty. \tag{17}$$

Relation (17) means that the first summand in the denominator of the lower bound estimate $\pi^{\text{estimate}(c)}_{\text{Esscher}(\alpha)}[X]$ is negligible with respect to the second summand in the denominator of the same estimate when $\alpha \to +\infty$. Using Markov inequality for the integral maximum, we get

$$\int_{\underline{b}}^{c} e^{\alpha x} dF_X(x) \; \le \; e^{\alpha c}\, \mathsf{P}\{X \le c\}. \tag{18}$$

It is also easy to see, due to the Markov inequality for the integral minimum, that for any $\varepsilon > 0$ such that $c + \varepsilon < \text{ess sup}[X]$ hold inequalities

$$\int_{c}^{\overline{b}} e^{\alpha x} dF_X(x) \; \ge \; \int_{c+\varepsilon}^{\overline{b}} e^{\alpha x} dF_X(x) \; \ge \; e^{\alpha(c+\varepsilon)}\mathsf{P}\{X \ge c + \varepsilon\} \; > \; 0. \tag{19}$$

Using inequalities (18) and (19), obtain

$$\frac{\int_{\underline{b}}^{c} e^{\alpha x} dF_X(x)}{\int_{c}^{\overline{b}} e^{\alpha x} dF_X(x)} \; \le \; \frac{\mathsf{P}\{X \le c\}\, e^{\alpha c}}{e^{\alpha(c+\varepsilon)}\mathsf{P}\{X \ge c+\varepsilon\}} \; \le \; \frac{\mathsf{P}\{X \le c\}}{\mathsf{P}\{X \ge c+\varepsilon\}}\, e^{-\alpha\varepsilon} \; = \text{constant} \cdot e^{-\alpha\varepsilon} \; \to \; 0, \quad \text{as} \quad \alpha \to +\infty,$$

and hence, limit relation (17) indeed holds. Dividing numerator and denominator of the lower bound estimate $\pi^{\text{estimate}(c)}_{\text{Esscher}(\alpha)}[X]$ by the integral $\int_{c}^{\overline{b}} e^{\alpha x} dF_X(x)$ and using limit relation (17), obtain

$$\pi^{\text{estimate}(c)}_{\text{Esscher}(\alpha)}[X] \; = \; \frac{\int_{c}^{\overline{b}} x e^{\alpha x} dF_X(x) \Big/ \int_{c}^{\overline{b}} e^{\alpha x} dF_X(x)}{\int_{\underline{b}}^{c} e^{\alpha x} dF_X(x) \Big/ \int_{c}^{\overline{b}} e^{\alpha x} dF_X(x) + 1} \; \sim \; \int_{c}^{\overline{b}} x e^{\alpha x} dF_X(x) \Big/ \int_{c}^{\overline{b}} e^{\alpha x} dF_X(x), \quad \text{as} \quad \alpha \to +\infty. \tag{20}$$

Observe also that

$$\int_{c}^{\overline{b}} x e^{\alpha x} dF_X(x) \Big/ \int_{c}^{\overline{b}} e^{\alpha x} dF_X(x) \; \ge \; c \cdot \int_{c}^{\overline{b}} e^{\alpha x} dF_X(x) \Big/ \int_{c}^{\overline{b}} e^{\alpha x} dF_X(x) \; = \; c. \tag{21}$$

From the limit relation (20) combined with the inequality (21) it follows

$$\lim_{\alpha \to +\infty} \pi_{\text{Esscher}(\alpha)}[X] \; \ge \; \lim_{\alpha \to +\infty} \pi^{\text{estimate}(c)}_{\text{Esscher}(\alpha)}[X] \; \ge \; c. \tag{22}$$

Combining inequality (22) with the limit variant of inequality (15), we finally get

$$\lim_{\alpha \to +\infty} \pi_{\text{Esscher}(\alpha)}[X] \; \in \bigcap_{\text{ess inf}[X] < c < \text{ess sup}[X]} [c, \; \text{ess sup}[X]] \; \equiv \; \{\text{ess sup}[X]\}.$$

This means that statement of Theorem 5.1 holds also in the case of a non-degenerated non-negative risk. □

Let us demonstrate an alternative way of proving of Theorem 5.1. This alternative proof is valid for the arbitrary choice of the random variable X, i.e., not only for the non-negative one.

Proof in the case of an arbitrary risk. Since we analyze limit behavior of the Esscher premium when the parameter α tends to plus infinity, then in all what follows, without of loss of generality, the parameter α can be treated as a strictly positive one. Let us chose a constant $c \in \mathbb{R}$ such that $c < \text{ess sup}[X]$. We will need the following set of transformations

$$\mathsf{E}[Xe^{\alpha X}] \;=\; \mathsf{E}[\mathbf{1}_{\{X\geq c\}}\, Xe^{\alpha X}] + \mathsf{E}[\mathbf{1}_{\{X<c\}}\, Xe^{\alpha X}] \;\geq\; c\,\mathsf{E}[\mathbf{1}_{\{X\geq c\}}\, e^{\alpha X}] + \mathsf{E}[\mathbf{1}_{\{X<c\}}\, Xe^{\alpha X}] \;=$$

$$= \; c\,(\mathsf{E}[e^{\alpha X}] - \mathsf{E}[\mathbf{1}_{\{X<c\}}\, e^{\alpha X}]) + \mathsf{E}[\mathbf{1}_{\{X<c\}}\, Xe^{\alpha X}] \;\geq\; c\,\mathsf{E}[e^{\alpha X}] - c\,e^{\alpha c} + \mathsf{E}[\mathbf{1}_{\{X<c\}}\, Xe^{\alpha X}]. \tag{23}$$

Let us show now that for any risk X holds the following limit relation

$$\frac{c\,e^{\alpha c}}{\mathsf{E}[e^{\alpha X}]} \;\to\; 0, \quad \text{as} \quad \alpha \to +\infty. \tag{24}$$

We chose $\varepsilon > 0$ such that $c + \varepsilon < \text{ess sup}[X]$, then, using inequality

$$\mathsf{E}[e^{\alpha X}] \;\geq\; e^{\alpha(c+\varepsilon)}\mathsf{P}\{X \geq c+\varepsilon\} \;>\; 0, \tag{25}$$

obtain

$$\left|\frac{c\,e^{\alpha c}}{\mathsf{E}[e^{\alpha X}]}\right| \;\leq\; \frac{|c|\,e^{\alpha c}}{e^{\alpha(c+\varepsilon)}\mathsf{P}\{X \geq c+\varepsilon\}} \;=\; \frac{|c|\,e^{-\alpha\varepsilon}}{\mathsf{P}\{X \geq c+\varepsilon\}} \;\to\; 0, \quad \text{as} \quad \alpha \to +\infty,$$

therefore, limit relation (24) indeed holds. Let us show now that for any risk X holds inequality

$$\lim_{\alpha\to+\infty} \frac{\mathsf{E}[\mathbf{1}_{\{X<c\}}\, Xe^{\alpha X}]}{\mathsf{E}[e^{\alpha X}]} \;\geq\; 0. \tag{26}$$

For this reason observe that the function $xe^{\alpha x}$ decreases on the interval $x \in (-\infty,\, -1/\alpha)$, increases on the interval $x \in (-1/\alpha,\, +\infty)$, achieves its minimum $-1/(\alpha e)$ at the point $x = -1/\alpha$, and takes negative values for the negative values of the variable x. While proving limit relation (26), we will consider separately case of $c \geq 0$ and $c < 0$. In the case of $c \geq 0$ we will majorette the function $\mathsf{E}[\mathbf{1}_{\{X<c\}}\, Xe^{\alpha X}]$ from below by the minimal value which the function $xe^{\alpha x}$ takes on $\mathbb{R}$, namely by $-1/(\alpha e)$. Then, using inequalities (25), we will get

$$\frac{\mathsf{E}[\mathbf{1}_{\{X<c\}}\, Xe^{\alpha X}]}{\mathsf{E}[e^{\alpha X}]} \;\geq\; \frac{-1/(\alpha e)}{e^{\alpha c}\mathsf{P}\{X \geq c\}} \;\to\; 0, \quad \text{as} \quad \alpha \to +\infty.$$

In the case of $c < 0$, for α large enough, namely for $\alpha \geq -1/c$, the function $xe^{\alpha x}$ will decrease on the interval $(-\infty,\, c)$ and will take its minimal value on the interval $(-\infty,\, c]$ at the point $x = c$. Therefore, the function $\mathsf{E}[\mathbf{1}_{\{X<c\}}\, Xe^{\alpha X}]$ on the mentioned interval can be constrained from below by the value $c\,e^{\alpha c}$. Using inequalities (25), we will get

$$\frac{\mathsf{E}[\mathbf{1}_{\{X<c\}}\, Xe^{\alpha X}]}{\mathsf{E}[e^{\alpha X}]} \;\geq\; \frac{c\,e^{\alpha c}}{e^{\alpha(c+\varepsilon)}\mathsf{P}\{X \geq c+\varepsilon\}} \;=\; \frac{ce^{-\alpha\varepsilon}}{\mathsf{P}\{X \geq c+\varepsilon\}} \;\to\; 0, \quad \text{as} \quad \alpha \to +\infty,$$

so as we see, limit relation (26) indeed holds for any admissible choice of the auxiliary parameter c. Combining inequalities (23) with limit relations (24) and (26), obtain

$$\lim_{\alpha\to+\infty} \pi_{\text{Esscher}(\alpha)}[X] \;=\; \lim_{\alpha\to+\infty} \frac{\mathsf{E}[Xe^{\alpha X}]}{\mathsf{E}[e^{\alpha X}]} \;\geq\; \lim_{\alpha\to+\infty} \left[c - \frac{c\,e^{\alpha c}}{\mathsf{E}[e^{\alpha X}]} + \frac{\mathsf{E}[\mathbf{1}_{\{X<c\}}\, Xe^{\alpha X}]}{\mathsf{E}[e^{\alpha X}]}\right] \;\geq\; c. \tag{27}$$

Switching to the limit in the inequality (15) and combining it with inequalities (27), we finally get

$$\lim_{\alpha\to+\infty} \pi_{\text{Esscher}(\alpha)}[X] \;\in\; \bigcap_{c<\text{ess sup}[X]} [c,\,\text{ess sup}[X]] \;\equiv\; \{\text{ess sup}[X]\}.$$

This completes the proof of Theorem 5.1 for an arbitrary choice of risk X. □

From inequality (15) it follows that the Esscher premium will converge to essential supremum of size of the insurance compensation *from below* as the premium parameter tends to plus infinity. The Esscher premium was defined only for the non-negative values of the parameter α. However, this is not out of scope of the article to show that for any random variable X holds the limit relation

$$\lim_{\alpha\to-\infty} \frac{\mathsf{E}[Xe^{\alpha X}]}{\mathsf{E}[e^{\alpha X}]} \;=\; \text{ess inf}[X]. \tag{28}$$

Let us show this. Using statement of Theorem 5.1, we will get

$$\lim_{\alpha\to-\infty} \frac{\mathsf{E}[Xe^{\alpha X}]}{\mathsf{E}[e^{\alpha X}]} \;=\; -\lim_{\alpha\to-\infty} \frac{\mathsf{E}[-Xe^{-\alpha(-X)}]}{\mathsf{E}[e^{-\alpha(-X)}]} \;=\; -\lim_{\beta\to+\infty} \frac{\mathsf{E}[-Xe^{\beta(-X)}]}{\mathsf{E}[e^{\beta(-X)}]} \;=\; -\,\text{ess sup}[-X] \;=\; \text{ess inf}[X],$$

therefore limit relation (28) indeed holds. The Esscher premium with the parameter α for a risk X is nothing else than just the expected value of the Esscher transform of the risk X, i.e., expectation of $Xe^{\alpha X} / \mathsf{E}[e^{\alpha X}]$, (however the Esscher transform was defined for $\alpha \in \mathbb{R}$ and not only for $\alpha \geq 0$ like it was in the case of the Esscher premium). The Esscher transform was introduced by the Swedish mathematician Fredrik Esscher in 1932, for the details see Esscher (1932) and Esscher (1963). Application of the Esscher transform to pricing of the insurance contracts probably was initiated by the Swiss mathematician Hans Bühlmann in the paper Bühlmann (1980). Description of the known results associated with the Esscher transform can be found, for example, in the reviewing paper by Yang (2004).

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